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# Heuristic Approach to Non-Abelian Quantum Kinematics and Dynamics in Configuration Spacetime

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New plausible kinematic foundations of quantum dynamics are discussed in a heuristic manner in which the quantum rule stems directly from the non-Abelian configuration symmetries of a system. Upon quantizing the 'complete' configuration symmetry group itself, irreducible generalized configuration-state representations can be calculated, whose transition amplitudes yield the propagation kernel. These states result from solving a set of 'generalized Schrödinger equations' corresponding to the superselection rules dictated by the quantized group. The propagation kernel of the system is thus obtained as an invariant Hurwitz integral, defined over the manifold of the complete symmetry group. A heuristic argument is given in favor of this approach to non-Abelian quantum kinematics, in which sums over physical world lines are evaluated instead of sums over arbitrary paths, for obtaining the propagation kernel of quantum systems having a classical Lagrangian analog. The attained quantum kinematic formalism, however, is completely general and does not depend on this particular interpretation. Nevertheless, the heuristic argument strongly suggests that non-Abelian quantum kinematics contains the formalism of standard nonrelativistic quantum mechanics as a very special case. No examples of the issues involved are presented in this paper.

## 1. INTRODUCTION

This paper concerns non-Abelian quantum kinematics and arose in the context of an undertaking to formulate quantum dynamics within the conceptual framework used in the quantum theory of symmetries (Krause, 1994a). The discovery of quantum mechanics is one of the greatest achievements of physics, but it is also one of the most difficult to grasp, even for those who have used it for decades. This circumstance makes it all the more necessary

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to explore the meaning of quantum mechanics from as many different perspectives as one may feel reasonable to conceive. Here we explore one of them, looking at a possible intimate connection between quantum mechanics and group theory.

The main idea pursued in this paper is the following. It seems that a compelling basis for having a new quantum formalism is provided by the extended role of *configuration space invariance* in the quantum description of a system. As we know, the superposition principle of quantum mechanics implies a different role for the configuration symmetries (as compared with the role they play in classical mechanics), since these symmetries are carried by the configuration states of the system in their linear unitary transformation laws. These states are thus classified as *irreducible representations* of the corresponding symmetry group. So it looks plausible to think of this fact as one of the most demanding features of quantum mechanics. The same idea has been considered recently, as a primary manifestation of symmetry, by Bohr and Ulfbeck (1995). (A new quantum formalism is proposed by these authors in which the probability laws originate in geometric correlations between group variables. However, they do not *quantize the group variables* as we do. This procedure separates two theories which, having many basic ideas in common, are radically different.)

There is now no *a fortiori* reason to demand the quantum rule to depend on some corresponding formal rule of classical mechanics, as it was unavoidable indeed 70 years ago (cf., Jammer, 1989). Strictly speaking, most physically relevant quantum systems known today have no classical analog at all. Hence, as a matter of principle, one should learn how to deal with them at the quantum level, relinquishing the correspondence principle. Moreover, physicists do now think *directly* in terms of *quantal notions*. Recent 'deductive interpretations' of quantum mechanics reveal this fact, making a strong contrast with the Copenhagen interpretation as to the status of classical physics (Omnès, 1994). It is rather clear that, if one puts aside the main conceptual working frames of classical mechanics, one may still rely on the *symmetry principles* and try to use them in a typical quantum-theoretic fashion without recourse to a classical analog. This is tantamount to considering the symmetry laws obeyed by quantum systems as the most demanding physical information concerning their mechanical structure and dynamical behavior.

Following this idea, it is the purpose of the present paper (a) to consider *generalized irreducible configuration-state representations* of a system, (b) to explore the *transition probability amplitudes* between such states, (c) to show that a satisfactory quantum formalism may emerge from *quantizing the configuration symmetry group* itself, and (d) to emphasize that *group quantization* provides the *irreducible set of fundamental variables* of the theory. We shall examine a theoretical framework within which such an

endeavor can be achieved. This formalism stems from an assessment of symmetry in quantum physics that differs substantially from the current views on symmetry prevailing in the literature (see, for instance, Fushchich and Nikitin, 1994).

In this fashion, the formulation of quantum dynamics introduced in this paper brings the *quantization rule* under a new group-theoretic perspective, which depends only on the symmetry laws of mechanics. Such a 'quantization-through-the-symmetry' program is conceivable because Lie groups can be 'quantized' quite generally by means of well-defined geometric procedures (Krause, 1991). This attempt entails a direct generalization of the Abelian kinematic standpoint underlying the standard formalism of quantum mechanics. As is well known, quantum kinematics was successfully initiated by Weyl (Weyl, 1931). Heisenberg's first paper on 'matrix mechanics' also addresses quantum kinematics (cf. van der Waerden, 1968). Weyl's most interesting achievement on this subject was his discussion of Heisenberg's kinematics as a unitary representation of the Abelian group of rigid translations in a Cartesian scaffolding. Indeed, his deduction of the fundamental commutation relations used in the canonical quantization procedure still remains as almost the best known contribution of (Abelian) quantum kinematics. On the other hand, Weyl's generalized non-Abelian group-theoretic quantization rule is not flexible enough for the purposes of physics, for it contains the fundamental commutators only *implicitly* (Daubechies, 1983). The interest of having a non-Abelian generalization of quantum kinematics emerges, for instance, as a required construct when one faces the problem of quantizing a system which is primarily described by noncommuting dynamical momenta satisfying a physically relevant non-Abelian Lie algebra (Kogut, 1980; Yamada, 1982).

Let us here remark also that in non-Abelian quantum kinematics as introduced in this paper it is not necessary to innovate in the foundation principles of quantum mechanics in general. These principles stand on firm physical ground, and we will innovate slightly by extending only *one* of them. Namely, the *time evolution postulate* (yielding the Schrödinger equation) will be here generalized into a *symmetry postulate* leading to a system of *invariant wave equations*, all standing on the same footing, within which the Schrödinger equation (if any) belongs (Krause, 1994a). In the current formalism of Hamiltonian quantum mechanics, *time* is the sole observable not represented by an operator in Hilbert space; rather, it enters the theory as a c-number parameter describing the evolution of wave functions. This rather peculiar feature has far-reaching consequences indeed. It means, for instance, that for the construction of quantum theories of spacetime, the choice of the time variable becomes a fundamental difficulty. In fact, an intriguing conflict thus arises between Hamiltonian quantum mechanics and the general covariance

of spacetime theories, such as the theory of general relativity, as is well known and has been discussed by several authors (Isham and Kushar, 1985; Hartle, 1988a,b, 1991; Gell-Mann and Hartle, 1992; Yamada and Takagi, 1991a,b, 1992) [for recent discussions on the 'problem of time' see also various articles in Ashtekar and Stachel (1991)]. It seems that, to circumvent these fundamental difficulties, a radical change in our approach to *quantization* may be in order.

Much remains to be said on this subject matter, to be sure. However, it is preferable to discuss the most general ideas and mathematical techniques entertained in non-Abelian quantum kinematics elsewhere, once we are in possession of some *physical intuition* on which the proposed formalism can be expected to be founded. So we devote this paper to giving only a cursory introduction to the essential ideas of our approach in a purely heuristic manner, with no pretension toward mathematical rigor or physical completeness. In this sense, we shall here concentrate on looking at a *physical motivation* for the present approach to quantum dynamics, which we believe deserves a special discussion by itself. It is hoped that the intuitive (albeit consistent) mathematics used in this paper will not distract the reader from the physical contents of the ideas. Indeed, this introduction should give an overall intuitive grasp of the subject matter in a synoptic setting.

The contents of this paper are as follows. Section 2 is devoted to studying symmetries and propagation kernels. (This study will lead us to an interesting heuristic problem.) In Section 2.1 we introduce the *new* concept of the 'complete symmetry group' of a system. In Section 2.2 the main features of 'sums over paths' are briefly summarized; these are then generalized into 'sums over world lines' in Section 2.3. We next explain this idea in Section 2.4, where we consider general superpositions of 'continuous' transition amplitudes, and in Section 2.5, which is a very sketchy discussion of the meaning of 'world line measurements' in quantum mechanics. In this fashion, relating both notions (i.e., 'complete symmetries' and 'sums over world lines'), we pose the problem tackled in this paper; namely, *to represent the propagation kernel as an invariant integral of world lines over the complete symmetry group* (Section 2.6). Section 3 is a synopsis of the main ideas on which the non-Abelian quantum kinematic foundations of quantum dynamics rest. (This approach yields a satisfactory, though heuristic, solution to the posed problem.) Quantum kinematics is rudimentarily addressed in Section 3.1, stressing the fundamental role played by superselection rules stemming from the 'complete' group. Section 3.2 is a short study of 'irreducible configuration ray representations' (which are in fact the heart of the theory). Configuration transition amplitudes and their relations to the theory of gauge symmetry in classical Lagrangian mechanics are discussed in Sections 3.3 and 3.4, respectively. (Here we touch again on Feynman's postulate, from which we

started.) Finally, Section 3.5 contains an intuitive discussion of the role played by quantum fluctuations and their possible connections with world-line symmetries. In Section 4 we present some concluding remarks.

For the sake of brevity we do not present examples of quantum kinematics in this paper, since these are always rather lengthy to work out (cf. Krause, 1986, 1988, 1995, 1996). Work is in progress concerning the quantum kinematic theory of the *complete symmetry group* of the classical Kepler system, which was introduced recently in the literature (Krause, 1994b). Other applications of the formalism will be considered elsewhere.

## 2. SYMMETRIES AND PROPAGATION KERNELS

In order to develop a quantization method based on the symmetry laws of dynamics, our starting idea is the concept of the *complete symmetry group* of a system (Krause, 1994b). We will review this idea for systems having a classical Lagrangian analog. On the other hand, the most outstanding result of the new quantization method is that it yields the *propagation kernel* as an invariant group integral that sweeps the whole manifold of allowable world lines (Krause, 1994a). Since this result is the consequence of the *complete action* of the group, it seems advisable to consider the possible *relations* between these two physical notions. The formal study of these relations will lead us to pose an interesting problem.

### 2.1. The Complete Symmetry Group of a System

We first examine the symmetry group. To this end, we need to introduce some notation. Henceforth, $G$ denotes a non-Abelian $r$-dimensional Lie group acting in the configuration space $X$ of a system. It will be sufficient to think of $X$ as a homogeneous space of $G$. In $X$ one has a preferred set of coordinates $x = (x^1, \ldots, x^n) \in X$, determined up to the transformations available in $G$, say

$$x'^{\mu} = f^{\mu}(x; q) \in X \tag{2.1}$$

$\mu = 1, \ldots, n$, where $q = (q^1, \ldots, q^r)$ denotes the parameters of the group. These transformations are kinematical automorphisms, which, by hypothesis, keep invariant the dynamical description of the system. Of course, they are endowed with the group property so that $f^{\mu}[f(x; q); q'] = f^{\mu}[x; g(q'; q)]$ holds. Here we have written

$$q''^{a} = g^{a}(q'; q) \in M(G) \tag{2.2}$$

with $a = 1, \ldots, r$, which entails the group multiplication law for the parameters on the group manifold $M(G)$. The point $e = (e^1, \ldots, e^r) \in M(G)$ labels the identity element of $G$, and we write $\bar{q} = \bar{q}(q) \in M(G)$ to denote that

unique point attached to $q$ which labels the inverse element of the element of $G$ labeled by $q$. Thus, one has $g^a[q; g(q'; q'')] = g^a[g(q; q'); q'']$ as well as $g^a(q; e) = g^a(e; q) = q^a$ and $g^a(q; \bar{q}) = g^a(\bar{q}; q) = e^a$. For the sake of simplicity, henceforth we assume that $G$ is a noncompact, connected, and simply connected Lie group, and that the $q$'s are real essential parameters maintaining everywhere a one-to-one correspondence with the elements of the group. [Concerning this rather strong assumption, see Krause (1991).] Note that the free and transitive action of $G$ in $X$ means that the diffeomorphisms $x \to x' = f(x; q)$ entail a *faithful* realization of the group. To proceed, all one needs to know is the faithful realization of $G$ in $X$. However, it will do just as well if the example of a *general configuration spacetime* (Trümper, 1983) is kept in mind for visualizing $X$, as we will do in what follows. Thus we write $x = (t, \mathbf{x}) \in X$.

Another question that needs some attention is the concept of symmetry itself. The notion of 'the *complete* symmetry group of a system' is more compelling for physics than the current notion of symmetry used hitherto. The main feature of this new concept is to characterize a system by the symmetry law it obeys, in a strictly specific manner (Krause, 1994b). One says that $G$ is a 'complete' symmetry group of a given system defined in $X$ if, and only if, the special realization of $G$: $x \to x' = f(x; q) \in X$ keeps invariant the equations of motion of the system, and *only* of that particular system. In classical mechanics, this means that the allowable world lines of the system become transformed into one another under the *complete* group of diffeomorphisms defining $G$ in $X$. In this sense, the *complete symmetry action* of the group contains implicitly all the essential information concerning the dynamical structure, and may be used as a faithful theoretical representative of the system itself. This notion demands the fulfilment of more tightened conditions than those required by the traditional Lie (and non-Lie) concepts of symmetry, which need not be specific to a given system. Let us mention here that, in general, the group realizations found in classical mechanics by the standard methods (see, e.g., Olver, 1986) fail to be 'complete.' For *linear* Newtonian systems, one can easily obtain the complete group operating in the manifold of their solutions (Aguirre and Krause, 1988a,b; Aguirre *et al.*, 1992). Whether such a specific realization of a complete symmetry group exists for any given *nonlinear* Newtonian system is not known (besides the Keplerian case).

## 2.2. Sum over Paths

Let us recall the standard situation in the Feynman path-integral approach (Feynman, 1948). We will here briefly recall the main features of this well-known general quantization procedure because there is a very natural way

of extending it (as a tool for calculating configuration transition amplitudes), on which we shall work presently.

In quantum mechanics, the main task is to calculate the propagation kernel. For closed systems, the propagator corresponds to the transition probability amplitude between two events in configuration spacetime. Since the configuration spacetime vectors $|x\rangle = |t, \mathbf{x}\rangle$ are equal-time complete, one may write $\langle x|x'\rangle$ as a 'sum over paths.' To include the contributions made by *all* paths, one must provide some *normalizing factor* in order to obtain a limit to the particular integration process involved. With this aim, Feynman introduced the following postulates:

(F.1) Each path $p(x; x') = (x \to x_1 \to \cdots \to x_{N-1} \to x') = \cup_{j=1}^{N} \Delta p_j$ contributes to $\langle x|x'\rangle$ with a *measure* $\mathcal{D}_N[p]$ which is the product of the local normalizing factors $\mathcal{D}[\Delta p_j]$ of the segments $\Delta p_j \subset p$ of that path.

(F.2) The *phase* $A[p]$ of the contribution made by a given path $p$ is the total classical action $A[p] = \Sigma_{j=1}^{N} A[\Delta p_j]$ of the system, evaluated over that particular path (in units of $\hbar$).

This means that the probability amplitude for the system to go from $x$ to $x'$, following a particular path $p(x; x')$, is given by the product of the probability amplitudes for the system to go through $x_{j-1}$ and $x_j$. Therefore, if *no* successive position measurements have been actually performed on the system (at times $t_1, t_2, \ldots, t_{n-1}$), one has

$$\langle x|x'\rangle_N = \sum_p \mathcal{D}_N[p(x; x')] e^{(i/\hbar)A[p(x;x')]}$$

$$= \sum_p \prod_{j=1}^{N} \mathcal{D}[\Delta p_j] e^{(i/\hbar)A[\Delta p_j]}|_{\Delta p_j \subset p} \tag{2.3}$$

which has a well-known meaning and yields a functional path integral when one takes the appropriate limit as $N \to \infty$. Thus one writes, in a symbolic notation,

$$\langle x|x'\rangle = \int \mathcal{D}[p(x; x')] e^{(i/\hbar)A_p(x;x')} \tag{2.4}$$

where $A_p(x; x') = A[p(x; x')]$ is the classical action of the system, evaluated along $p(x; x')$ between the extreme points $x$ and $x'$. The 'measure' $\mathcal{D}[p(x; x')]$ is in general complex and depends on the dynamics. These things are, of course, well known. [An outline of the general theory of path integrals in quantum mechanics and how to solve them, reflecting the progress made during recent years, can be found, for instance, in Grosche and Steiner, (1995).]

## 2.3. Sum over World Lines

Now we can begin our understanding of the new quantum kinematic formalism with the consideration of a system which has a classical analog evolving along well-defined world lines in $X$. In the sequel we assume that equation (2.1) implies that $G$ acts as the *complete* symmetry group of the system. We next examine some changes in physical outlook which may be introduced by this strong notion of symmetry in quantum mechanics.

The following remark is made by Feynman and Hibbs (1965): "It is possible to define the path in a somewhat more elegant manner. Instead of straight lines between the points $j$ and $j + 1$, we could use sections of the classical orbit. Then we could say that $A$ is the minimum value of the integral of the Lagrangian over all the paths which go through the specified points $x_j = (t_j, \mathbf{x}_j)$. With this definition no recourse is made to arbitrary straight lines." (They elaborate a bit on this idea, producing a mathematical technique that is very useful for computing the sum over paths in some rather special situations.)

Let us pursue this idea under a completely general perspective. First, it means that to define a subset of all possible paths, instead of using straight lines $\Delta p_j = x_j - x_{j-1}$, we could use world-line sections pertaining to that particular world line $w = w(x_{j-1}; x_j)$ which goes through the points $x_{j-1}$ and $x_j$; i.e., we can approximate $\Delta p_j \approx \Delta w_j$ and $A[\Delta p_j] \approx A[\Delta w_j]$ everywhere on each chosen subset of paths. Second, this idea requires the introduction of a new local normalizing factor $\mathcal{D}[\Delta p_j] \approx \mathcal{W}[x, x'; \Delta w_j]$ such that one has

$$\mathcal{D}[\Delta p_j]e^{(i/\hbar)A[\Delta p_j]} = \mathcal{W}[x, x'; \Delta w_j]e^{(i/\hbar)A[\Delta w_j]}|_{\Delta w_j \approx \Delta p_j} \tag{2.5}$$

Clearly, the local normalizing factor $\mathcal{W}$ must be now a function of the extreme points $x$ and $x'$, because in general $\Delta w_j \subset w(x_{j-1}; x_j) \neq w_e(x; x')$, where $w_e$ is that *unique* world line which goes through $x$ and $x'$. Hence, one interprets $\mathcal{W}[x, x'; \Delta w_j] \exp\{(i/\hbar)A[\Delta w_j]\}$ as the conditional probability amplitude for the system to have gone from $x_{j-1}$ to $x_j$ obeying its own *law of motion*, given the total transition $x \to x'$. If one draws a picture for visualizing this peculiar idea, one sees that *all* the allowable world lines of the system make a contribution to the total transition probability amplitude $\langle x | x' \rangle$.

But then we can change the physical look of the problem. Instead of thinking of an imaginary experiment in which successive *position measurements* could be performed on the system (at times $t_1, t_2, \ldots, t_{N-1}$), one rather thinks of an imaginary experiment in which one could determine if the system has evolved on a *given world line* $w$, either at the intervals $t_1 - t$, $t_2 - t_1$, $\ldots$ or $t' - t_{N-1}$. Note that the apparatus setup used in these two kinds of 'gedanken' experiments are completely different. (The mechanical principles invoked in the second experiment are very simple; cf. below.) With an

experiment of this kind in mind, according to the laws of quantum mechanics, one has

$$\mathcal{W}_N[x, x'; w]e^{(i/\hbar)A[w]} = \prod_{j=1}^{N} \mathcal{W}[x, x'; \Delta w_j]e^{(i/\hbar)A[\Delta w_j]}|_{\Delta w_j \subset w} \tag{2.6}$$

which yields the probability amplitude for the system to go from event $x$ to $x'$, having 'moved' on a particular world line $w$ *sometime* between $t$ and $t'$. Therefore, if *no* experiment is performed at all, one calculates the configuration transition amplitude $\langle x|x'\rangle_N$ as a 'sum over world lines'; namely, one sets (symbolically)

$$\langle x|x'\rangle_N = \sum_w \mathcal{W}_N[x, x'; w]e^{(i/\hbar)A[w]} \tag{2.7}$$

instead of equation (2.3). In fact, equations (2.3) and (2.7) are two examples of the general rule of superposition of probability amplitudes in quantum mechanics.

## 2.4. General Superposition of Continuous Probability Amplitudes

To understand further the physics on which the heuristic argument rests, let us provide some explanations for the two basic notions used in a 'sum over world lines.' We first recall the main ideas concerning amplitude superpositions in quantum mechanics for the important case of *continuous* spectra. ('World-line measurements' are briefly considered in the next section.)

Let us look at a more general formula to describe transition amplitudes, which still contains the essential features of a Feynman path integral. Let the $\mathbf{x}$'s be Cartesian coordinates and let $\mathbf{B}$ denote a complete set of compatible time-independent observables in the Schrödinger picture of the system, such that $[\mathbf{X}, \mathbf{B}] \neq 0$. We write, rather sketchily, $\mathbf{B}|\mathbf{b}\rangle = \mathbf{b}|\mathbf{b}\rangle$ and consider that the spectrum of $\mathbf{B}$ is continuous. We next introduce the Heisenberg picture: $|\mathbf{b}\rangle \to |b\rangle = |t, \mathbf{b}\rangle$ and $\mathbf{B} \to \mathbf{B}(t)$. Thus, the $|b\rangle$'s (like the $|x\rangle$'s) are only equal-time complete and equal-time orthogonal.

Our first aim is to calculate a transition amplitude $\langle x|x'\rangle_N$ upon the condition that the following $N$-step experiment has been performed on the system:

$$\mathbf{X}_t \to |x\rangle \to \mathbf{B}_{(1)} \to |b_1\rangle \to \cdots \to |b_{N-1}\rangle \to \mathbf{B}_{(N)} \to |b_N\rangle \to \mathbf{X}_{t'} \to |x'\rangle$$

whose meaning is clear: $\mathbf{X}_t$ is an initial preparation of configuration state; then, the system interacts successively $N$ times with a "$\mathbf{B}$-meter" (which is set "on" only at times $t_1 < t_2 < \cdots < t_N$, with $t < t_1$ and $t_N < t'$); finally, $\mathbf{X}_{t'}$ denotes a detection of the outcoming configuration state. It might be argued that such an idealized experiment borders on fiction; actually, it

does so no more than those thought experiments currently used in quantum mechanics. We define a '$b$-path' $p(b;\, b') = (b = b_1 \to b_2 \to \cdots \to b_{N-1} \to b_N = b')$ for each possible outcome of the $N$-step experiment, so that in every actual run of the apparatus one has a probability-density amplitude given by

$$\varphi_{x,p(b;b'),x'} = \langle x \mid b_1\rangle\langle b_1 \mid b_2\rangle \cdots \langle b_{N-1} \mid b_N\rangle\langle b_N \mid x'\rangle \tag{2.8}$$

Upon the *actual* performance of such experiment, the conditional probability density for observing a particular path $p(b;\, b')$ (given the total observed transition $x \to x'$) would read $P_N[p(b;\, b') \mid (x,\, x')] = |\varphi_{x,p(b;b'),x'}|^2$, whence one obtains the total conditional probability $P_N(x \mid x')$ (for the transition $x \to x'$) by integrating this expression $N$ times over the whole spectrum of the observable **B**.

However, if *no measurements are made* between $x$ and $x'$, in quantum mechanics one has the well-known and remarkable law (Feynman, 1948)

$$\begin{aligned}
&\langle x \mid x'\rangle_N \\
&\quad = \int_{\mathbf{b}_1} \cdots \int_{\mathbf{b}_N} d(\mathbf{b}_1) \cdots d(\mathbf{b}_N)\, \langle x \mid b_1\rangle\langle b_1 \mid b_2\rangle \cdots \langle b_{N-1} \mid b_N\rangle\langle b_N \mid x'\rangle \\
&\quad = \prod_{j=1}^{N} \int_{\mathbf{b}_j} d(\mathbf{b}_j)\psi_{b_1}(x) \prod_{k=1}^{N-1} \langle b_k \mid b_{k+1}\rangle \psi^*_{b_N}(x') \\
&\quad = \sum_p \psi_b(x)\mathfrak{D}_N[x,\, x';\, p(b;\, b')]\psi^*_{b'}(x')e^{(i/\hbar)\Phi[p(b;b')]}
\end{aligned} \tag{2.9}$$

This has been written as a 'generalized sum over $b$-paths.' To this end, one introduces a suitable notation. (1) two 'wave functions'

$$\psi_b(x) = \langle x \mid b_1\rangle, \qquad \psi_{b'}(x') = \langle x' \mid b_N\rangle \tag{2.10}$$

which are defined at the extreme configurations; (2) the 'local contributions,' say

$$\mathfrak{D}[x,\, x';\, \Delta b_k]e^{(i/\hbar)\Phi[\Delta b_k]} = \langle b_k \mid b_{k+1}\rangle \tag{2.11}$$

made by each intermediate transition amplitude in going from $x$ to $x'$; (3) the 'total contribution' of a given path $p(b;\, b')$,

$$\mathfrak{D}_N[x,\, x';\, p(b;\, b')]e^{(i/\hbar)\Phi[p(b;b')]} = \prod_{k=1}^{N-1} \langle b_k \mid b_{k+1}\rangle \tag{2.12}$$

given by the product of the intermediate transition amplitudes along that path (as it must be); and finally, (4) the symbol denoting the 'sum over paths'

$$\sum_{p} \equiv \prod_{j=1}^{N} \int_{\mathbf{b}_j} d(\mathbf{b}_j) \tag{2.13}$$

which is just a shorthand notation.

In this manner, the continuous limit of equation (2.9) must be invoked here, since the 'gedanken' experiment is now replaced by purely *virtual* processes. Thus, taking $N \to \infty$, one writes the *general functional path integral* for the desired transition probability amplitude in the following fashion:

$$\langle x \,|\, x' \rangle_{(b;b')} = \int \mathcal{D}[x, x'; p(b; b')] \psi_b(x) \psi_{b'}^{*}(x') e^{(i/\hbar)\Phi[p(b;b')]} \tag{2.14}$$

where one must integrate over all continuous paths $p(b; b')$ between two *given limit values* $\mathbf{b}$ and $\mathbf{b}'$ of $\mathbf{B}$ at $t$ and $t'$. One notices the need to define two wave functions $\psi_b(x) = \langle x \,|\, b \rangle$ and $\psi_{b'}(x') = \langle x' \,|\, b' \rangle$ at the extreme points, which result from the limiting process. In fact, equation (2.14) entails a *functional correlation* between these two wave functions. For instance, if $\mathbf{B} = \mathbf{P} =$ (linear momentum operators), then one has two correlated plane waves at the extremes: $\psi_p(x) = \exp[(i/\hbar)\mathbf{x} \cdot \mathbf{p}]$ and $\psi_{p'}(x') = \exp[(i/\hbar)\mathbf{x}' \cdot \mathbf{p}']$. In this approach, the real task for quantum dynamics is to provide a sensible meaning for the local transition amplitudes defined in equation (2.11); these are the conditional probability amplitudes for the system to make the 'virtual quantum jumps' $b_k \to b_{k+1}$ while actually evolving *freely* from $x$ to $x'$ according to its own dynamical structure.

Note that one may also think of equation (2.14) as exhibiting a kind of 'decoherence effect' of the system's wave function as it evolves *freely* (that is, obeying its own dynamical laws), while suffering quantum fluctuations owing to the environment (see Omnès, 1992, and references quoted therein). Notwithstanding the fact that the functional integral shown in equation (2.14) is not in the usual form of a 'decoherent' transition amplitude, the underlying ideas seem to be very close. We just mention this plausible conjecture here, for it might deserve a careful study.

Notwithstanding all its mathematical drawbacks, the general expression arrived at in equation (2.14) symbolizes a correct quantum mechanical result. The physical content of such integrals is independent of their mathematical awkwardness. As a matter of fact, Feynman's path integrals in quantum mechanics are a special instance of equation (2.14). What makes them simpler (and more manageable, up to a point) is the fact that in the Feynman spacetime approach to quantum mechanics one uses $|\, b \rangle = |\, x \rangle = |\, t, \mathbf{x} \rangle$ and one makes a physically sound choice at the step of equation (2.11). However this may

be, we observe that the 'sum over world-lines' stated in equation (2.7) is also a particular case of equation (2.9); that is, before one takes the very critical limit $N \to \infty$.

## 2.5. World-Line Measurements in Quantum Mechanics

We have not yet discussed the principles underlying the performance of a *world-line measurement*, which is the other conceptual ingredient leading to equation (2.7). To make things easy, let us consider a Newtonian system.

The world lines depend on a set of constants of integration, say $\{\alpha, \beta\}$, which may be subjected to arbitrary continuous variations. Thus, one may consider the world lines themselves as the set of 'primitive curves' $W_{CL}(t, \mathbf{x}; \alpha, \beta) = 0$, with 'tangents' given by $\dot{W}_{CL} = 0$. In this way, the differential equations of motion are just the 'eliminants' of $\alpha$ and $\beta$ between $W_{CL} = 0$ and $\dot{W}_{CL} = 0$. This means, however, that one has $\alpha = \mathbf{A}[t, \mathbf{x}(t), \dot{\mathbf{x}}(t)]$ and $\beta = \mathbf{B}[t, \mathbf{x}(t), \dot{\mathbf{x}}(t)]$, quite generally. These are *basic* constants of motion, such that all *mechanical constants of motion* (such as energy or some momenta) are functions thereof, and vice versa. Therefore, a 'world-line measurement' in classical mechanics means that one determines a well-defined history of a system in $X$ by measuring *enough* mechanical constants of motion in a given state of the system. (It is not strictly necessary to measure simultaneously $\mathbf{x}$ and $\dot{\mathbf{x}}$ in order to determine a world line.)

Of course, owing to quantum fluctuations, there are no world lines in quantum mechanics. Because of the same fact, there are no paths either. However, it is rather clear that the analog of a 'world-line measurement' in quantum mechanics corresponds precisely to the simultaneous measurement of a *complete set of compatible constants of motion* in a given state of the system. This fact brings to the fore the conservation laws and superselection rules obeyed by the system; namely, the *symmetry laws* of quantum mechanics.

## 2.6. The Propagation Kernel as an Invariant Group-Integral

Notwithstanding these explanations, it seems that the idea of changing the 'sum over all paths' into a 'sum over all world lines,' albeit elegant and formally conceivable, raises more problems than it can actually solve. Nonetheless, it is worth facing these problems.

The first problem is that one needs a *continuous parametrization* to label the allowable world lines of the classical analog so that, in the limit $N \to \infty$, equation (2.7) yields a well-defined integral. It is immediate that the best (if not the only) way of tackling this problem is to label the world lines as $w_q$ by means of the parameters $q = (q^1, \ldots, q^r)$ of the complete symmetry group $G$ of the system. Thus we write, for any given world line

$$w_q(x; x') = w_e[f(x; q); f(x'; q)] \tag{2.15}$$

where $w_e(x; x')$ denotes the classical history between the extreme points $x$ and $x'$ and $f(x; q)$ is given in equation (2.1). Bear in mind that the mappings $w_e(x; x') \to w_q(x; x')$ correspond to an exhaustive group of diffeomorphisms acting over the whole world-line manifold, which is indeed isomorphic with $G$, for any given pair of events $(x, x')$. Furthermore, since the action of $G$ is *complete* in $X$, no 'extraneous' world lines are obtained by means of these mappings.

In this fashion, given a complete symmetry group, we are in position to conceive a limit $N \to \infty$ to equation (2.7). Thus we expect to be able to write the propagation kernel as a group-theoretic integral of the general form

$$\langle x | x' \rangle = \int d\mu(q)\ W(x, x'; q) e^{(i/\hbar)A_q(x;x')} \tag{2.16}$$

This integral has essentially the *same physical contents* as the corresponding Feynman functional path integral. Observe that in equation (2.16) it is enough to require that $d\mu(q)$ be the *Hurwitz left-invariant measure* of $G$ in $M(G)$. In this way, as $q$ sweeps the group manifold, the integral picks up the contributions made by all world lines $w_q(x; x')$, with a 'measure density' $W(x, x'; q)$ defined in $M(G)$, and with a phase function given by

$$A_q(x; x') = A[f(x; q); f(x'; q)] \tag{2.17}$$

in units of $\hbar$, as it must be according to Feynman's postulate, whence these ideas stem.

In summary, within the heuristic approach leading to equation (2.16), we see that the only problem left is to find the appropriate measure density $W(x, x'; q)$. This problem belongs to the theory of symmetries in quantum mechanics. Notwithstanding the fact that this is a fundamental question on the system dynamics, if the action of $G$ is *complete* in $X$, this question becomes automatically reduced to a problem in quantum kinematics. This issue states our program, for it is in this sense that we search for a theory that, in the last analysis, entails a *geometrization of quantum dynamics.*

## 3. SYNOPSIS: QUANTUM KINEMATICS AND DYNAMICS

We devote this part of the paper to a solution to the heuristic problem posed in the previous sections. Like the problem itself, the proposed solution is suggested by our previous work [especially Krause (1986, 1988)]. Due to considerations of space, our presentation will be rather schematic.

### 3.1. Non-Abelian Quantum Kinematics

In order to find such a theory within which the integral stated in equation (2.16) can be actually evaluated, we concentrate on the bracket that figures

in the left-hand side of that equation. The reason for this is the following. Lie group theory affords a nontrivial resolution of the identity, i.e.,

$$I = \int d\mu(q)\, |q\rangle\langle q| \tag{3.1}$$

The Lie group resolution of the identity is completely general and independent of the physical states of the system. It is enough to notice here that the vectors $|q\rangle$ belong to the continuous orthogonal complete basis of the *rigged* Hilbert space $\mathcal{H}(G)$ that carries the left regular representation of $G$ (Krause, 1991, 1994a). They satisfy the covariant transformation law

$$U_k(q')\,|q\rangle = e^{(i/\hbar)\phi_k(q';q)}\,|g(q';\, q)\rangle \tag{3.2}$$

where $U_k(q) = \exp\{-(i/\hbar)q^a L_a^{(k)}\}$ denotes the representative operators and $\phi_k(q';\, q)$ is a two-cocycle, belonging to a *central extension* by $U(1)$ of the *left regular representation* of the group (Krause, 1987). The generators $L_a^{(k)}$ satisfy the well-known extended Lie algebra (Bargmann, 1954). In fact, unitary *ray* representations must be used in quantum kinematics as far as possible.

Starting from the fact that $G$ is a *complete* symmetry group acting in $X$, quantum kinematics is able to produce a quantum model that meets all the essential features required in equation (2.16). This calls for many mathematical refinements to be taken up elsewhere. Nonetheless, the procedure leading to this achievement is conceptually simple. It consists mainly in two steps:

(QK.1) One first *quantizes the group* $G$ itself, on the group manifold $M(G)$.

(QK.2) One next finds *irreducible configuration representations* of $G$ in $X$.

Using this conceptual scheme, one arrives at a reasonable solution to the problem stated above. We next sketch these two steps.

The first step (QK.1) means that one introduces a complete set of *generalized position operators* $q^a \to Q^a = Q^{a\dagger}$ on the group manifold ($a = 1, \ldots, r$) such that $Q^a\,|q\rangle = q^a\,|q\rangle$ holds. These operators obey the following covariant law under the action of the group:

$$U_k^\dagger(q) Q^a U_k(q) = g^a(q;\, Q) = \int d\mu(q')\, |q'\rangle g^a(q;\, q')\langle q'| \tag{3.3}$$

Therefore, as an immediate consequence of this law, they satisfy *generalized Heisenberg commutation relations* with the generators of the (left) regular ray representation (Krause, 1985):

$$[Q^a,\, L_b^{(k)}] = i\hbar R_b^a(Q) \tag{3.4}$$

Here $R^a_b(q) = \lim_{q' \to e} \partial'_b g^a(q'; q)$ stands for the right transport matrix for contravariant vectors in $M(G)$. Such *explicit* commutation relations are of potential value for physics. In particular, if $G$ is the Abelian group of space translations (and the parameters are canonical), equations (3.4) become the standard Heisenberg commutation relations. In this sense, it seems interesting to remark that there is *no way* to arrive at the generalized commutators (3.4) by means of the usual canonical quantization procedure.

It must be borne in mind that the set of operators $\{Q^1, \ldots, Q^r; L^{(k)}_1, \ldots, L^{(k)}_r\}$, and not just the set of generators $\{L^{(k)}_1, \ldots, L^{(k)}_r\}$, affords the *irreducible set of basic observables* characterizing the (rigged) Hilbert space $\mathcal{H}(G)$. Indeed, for a *complete* symmetry group, it turns out as a necessary condition that *all dynamical variables* of the theory are functions of the $Q$'s and the $L$'s (Krause, 1994a). One should also notice that the rigged Hilbert space that carries the quantum kinematic models is provided by the *dynamically complete* group itself.

From the standpoint of our present interest, the main reason for defining position operators in $M(G)$ is that equations (3.4) allow us to obtain a *maximal set of compatible superselection rule operators* $S_\alpha = S^\dagger_\alpha$, $[S_\alpha, S_\beta] = 0$, with $\alpha, \beta = 1, \ldots, s < r$. This set is in fact *larger* than the traditional set of Casimir operators, which is all one has when one does not quantize the group, and includes them as a *proper* subset (Krause, 1991, 1993a). For any given noncompact non-Abelian Lie group $G$ the operators $S_\alpha = S_\alpha(Q; L^{(k)})$ are *known* functions of the $Q$'s and the $L$'s which commute with all the $L$'s: $[S_\alpha, L^{(k)}_a] = 0$. Thus, one has

$$U^\dagger_k(q) S_\alpha U_k(q) = S_\alpha \tag{3.5}$$

for all $q \in M(G)$. Hence, if one solves the simultaneous eigenvalue equations

$$S_\alpha |\psi_\epsilon\rangle = \epsilon_\alpha |\psi_\epsilon\rangle \tag{3.6}$$

one obtains a *maximal* decomposition of the regular representation into *irreducible representations* carried by the eigenvectors $|\psi_\epsilon\rangle \in \mathcal{H}_\epsilon$, where each $\mathcal{H}_\epsilon$ is an invariant subspace of $\mathcal{H}(G)$.

One then introduces a very natural *postulate* in order for physics to make contact with group theory; namely, one requires that *the only allowable physical states of the system are eigenvectors of the superselection rule operators* (Krause, 1994a). This makes $\mathcal{H}(G)$ into an *incoherent* Hilbert space carrying the irreducible physical models of the system. Moreover, in this fashion, if $X$ corresponds to an 'external' configuration space, an interesting *isotopic structure* comes to the fore in its own right [for details, see Krause (1993b)].

### 3.2. Irreducible Configuration Representations

Accordingly, from this point of view, the fundamental problem consists essentially in finding configuration representations of the complete action of $G$ in $X$, carried by vectors $|x\rangle \leftrightarrow x \in X$, $|x\rangle \in \mathcal{H}(G)$, such that something like

$$\langle x|x'\rangle = \int d\mu(q)\, \langle x|q\rangle\langle q|x'\rangle = \int d\mu(q)\, W(x, x'; q)e^{(i/\hbar)A_q(x;x')} \quad (3.7)$$

can be expected to hold.

We then come to the fundamental 'dynamical' step (QK.2) (which, however, is also pure kinematics). In our previous work (Krause, 1994a) it is proved that, within the general formalism developed in step (QK.1), one can calculate physical configuration states $|x; \epsilon\rangle = |x^1, \ldots, x^n; \epsilon_1, \ldots, \epsilon_s\rangle$ which are in one-to-one correspondence with the points $x \in X$. Such states carry a *configuration ray representation*, since they obey the following *covariant kinematic law* with respect to the complete action of $G$ in $X$:

$$U_k(q)|x; \epsilon\rangle = e^{(i/\hbar)\varphi_k(x;q)}|f(x; q); \epsilon\rangle \quad (3.8)$$

The exponent function $\varphi_k(x; q)$ is a local real phase, stemming from the central ray representation of $G$, and may be calculated once the functions $\phi_k(q'; q)$ and $f(x; q)$ are known. These vectors describe *physical states* of the system evolving in configuration spacetime, for they also satisfy the superselection rules

$$S_\alpha|x; \epsilon\rangle = \epsilon_\alpha|x; \epsilon\rangle \quad (3.9)$$

Therefore, they carry an *irreducible configuration ray representation* of the complete action of $G$ in $X$, i.e., $|x; \epsilon\rangle \in \mathcal{H}_\epsilon$.

Clearly, to obtain these irreducible configuration states is the main task in the quantum kinematic approach to dynamics. This is achieved in the following fashion. The *known* functional form of the superselection operators $S_\alpha(Q; L^{(k)})$ as well as the definition of the associated wave function in configuration spacetime

$$\psi_\epsilon(x) = \lim_{q\to e} \langle x; \epsilon|q\rangle = \langle x; \epsilon|e\rangle \quad (3.10)$$

allow one to write in $X$ the system of *simultaneous wave equations* corresponding to the superselection rule equations (3.9):

$$\mathcal{S}_\alpha^{(k)}(x)\psi_\epsilon(x) = \epsilon_\alpha\psi_\epsilon(x) \quad (3.11)$$

The form of the 'scalar operators' $\mathcal{S}_\alpha^{(k)}(x) \leftrightarrow S_\alpha(Q; L^{(k)})$, which are partial differential operators acting in $X$, is calculated in the usual manner (see, e.g., Cornwell, 1989). We call these equations in the system of *generalized*

*Schrödinger equations*. They play the fundamental role in the theory, for solving them yields the irreducible configuration vectors, which read

$$|x;\epsilon\rangle = \int d\mu(q)\ \psi_\epsilon^*[f(x;\bar{q})]e^{(i/\hbar)\varphi_k(x;\bar{q})}|q\rangle \tag{3.12}$$

These configuration states describe a quantum dynamical model of the system for which the action of $G$ is complete in $X$, and *only* of that particular system. It must be underlined here that no 'generalized configuration operators' (i.e., $x^\mu \to X^\mu$) have been used to this end. In other words, in the kinematic approach to dynamics one does not need to 'quantize' the configuration spacetime coordinates $x = (x^1, \ldots, x^n)$; they *all* appear on the same footing, as c-number parameters labeling the physical states of the system, as required by the 'relativity theory' of $G$ in $X$ (Mariwalla, 1975). In this sense, the present approach is in fact more akin to quantum field theory.

Furthermore, for an *isolated system* time translation is a symmetry transformation and therefore the usual *Schrödinger equation* is obtained automatically as *one* of the generalized wave equations (3.11) stemming from $t \to t' = t + q^0$. This means that the *known* functional form of the associated superselection operator, say $S_0^{(k)}(t, \mathbf{x}) \leftrightarrow S_0(Q; L^{(k)})$, once substituted into (3.11), allows one the *explicit calculation of the Hamiltonian operator* of the system, within the quantum kinematic model itself, without recourse to a prequantized classical analog. [How this approach actually works can be seen in the examples presented in Krause (1986, 1988, 1996).]

### 3.3. Configuration Transition Amplitudes

Equations (3.11) and (3.12) are in fact the keys leading to quantum dynamics, because the desired transition amplitudes follow immediately. Indeed, from (3.1) and (3.2) one obtains

$$\begin{aligned}
&\langle x;\epsilon|x';\epsilon'\rangle \\
&\quad = \int d\mu(q)\ \langle x;\epsilon|q\rangle\langle q|x';\epsilon'\rangle \\
&\quad = \int d\mu(q)\ \langle x;\epsilon|U_k(q)|e\rangle\langle e|U_k^\dagger(q)|x';\epsilon'\rangle \\
&\quad = \int d\mu(q)\ \langle f(x;\bar{q});\epsilon|e\rangle\langle e|f(x';\bar{q});\epsilon'\rangle e^{(i/\hbar)[\varphi_k(x';\bar{q})-\varphi_k(x;\bar{q})]}
\end{aligned} \tag{3.13}$$

where we have used the property $U_k(q)|e\rangle = |q\rangle$. Thus one ends with an invariant integral over the group manifold; i.e.,

$$\langle x;\epsilon|x';\epsilon'\rangle = \int d\mu(q)\ \psi_\epsilon^*[f(x;\bar{q})]\psi_{\epsilon'}[f(x';\bar{q})]e^{(i/\hbar)[\varphi_k(x';\bar{q})-\varphi_k(x;\bar{q})]} \tag{3.14}$$

which yields the desired transition amplitudes between the irreducible configuration eigenstates corresponding to the system of generalized Schrödinger equations.

This result has the required form shown in the general formula (2.14). The 'general sum over paths' has been here evaluated as a 'sum over $q$' [$q \in M(G)$]; in this way one ensures a 'sum over all world lines.' We also note that the problem set by the 'two wave functions' which must be defined at the extreme configuration points $(x, x')$ becomes solved in quantum kinematics in a precise manner.

Let us here only remark that the transition amplitude (3.14) must be consistent with the orthogonal property $(\epsilon_\alpha - \epsilon'_\alpha)\langle x; \epsilon | x'; \epsilon' \rangle = 0$, which follows from (3.9). This means that in evaluating this integral one necessarily obtains a delta function $\delta^{(s)}(\epsilon - \epsilon')$, which factorizes out as a consequence of the superselection rules. (We have here assumed continuous spectra for all the superselection operators.) This is a peculiar property of the present approach; it shows that every quantum kinematic model calculated along these lines yields a theory which is already 'regular.'

### 3.4. Gauge Symmetries in Lagrangian Mechanics

Hence, the question arises as to what are the possible relations between the sum-over-world-line transition amplitude (2.16) and the quantum-kinematic transition amplitude shown in (3.14). Interestingly enough, the answer to this question comes from classical mechanics.

Several years ago, Lévy-Leblond (1964) discussed the Lagrangian gauge problem from the standpoint of the group-theoretic foundations of classical mechanics. The starting point of his analysis is the action integral $A(x; x')$ of the system. He proceeds to study the conditions for the equations of motion to be invariant under a non-Abelian noncompact Lie group $G$ of kinematic automorphisms, acting freely and transitively in $X$, as in equation (2.1). In this fashion, Lévy-Leblond explores the consequences of the following gauge transformation of the action:

$$A[f(x; q); f(x'; q)] = A(x; x') + \varphi_k(x'; q) - \varphi_k(x; q) \qquad (3.15)$$

induced by the elements of $G$, which keeps the Euler–Lagrange equations invariant. With these assumptions, Levy-Leblond shows that there is enough information to calculate the allowable 'classical gauge functions' $\varphi_k(x; q)$. Thus, he develops the mathematical formalism to this end, which depends exclusively on the structure of $G$ and on the functions $f(x; q)$ describing the action of the group in configuration spacetime.

Now, it turns out that Levy-Leblond's *classical method* for obtaining the gauge function $\varphi_k(x; q)$ that appears in (3.15) is *exactly* the same as the

*quantum kinematic method* for obtaining the exponent function $\varphi_k(x; q)$, that figures in (3.8) [which is developed in Krause (1994a)], i.e., both functions are just *equal*, and they are defined up to the same gauge freedom.

Hence, we can substitute from (3.15) into (3.14) quite rigorously, and therefore the quantum-kinematic configuration transition amplitude of a Lagrangian system is given by

$$\langle x; \epsilon \mid x'; \epsilon' \rangle = \int d\mu(q)\, \psi_\epsilon^*[f(x; \bar{q})]\psi_{\epsilon'}[f(x'; \bar{q})]e^{(i/\hbar)[A_{\bar{q}}(x;x') - A(x;x')]} \tag{3.16}$$

which has the expected form (2.16) of a 'sum over world lines.' In fact, the particular case of a quantum system which has a classical Lagrangian analog becomes described by introducing the following *measure density* over the group manifold:

$$W_{\epsilon\epsilon'}(x'x'; q) = \psi_\epsilon^*[f(x; \bar{q})]\psi_{\epsilon'}[f(x'; \bar{q})]e^{-(i/\hbar)A(x;x')} \tag{3.17}$$

in order to embrace the world-line manifold of the system.

An interesting feature appearing in (3.16) is the difference in the action $A_{\bar{q}}(x; x') - A(x; x')$. As we see, in quantum kinematics, it is *not* just the value of the action over the transformed world line itself [i.e., $A_{\bar{q}}(x; x')$] that yields the *effective physical phase* of the world-line contributions to the transition probability amplitude; rather, it is the *difference* with the value of the corresponding *classical action* $A(x; x')$ that does the job.

## 3.5. Why World Lines?

Since the Feynman approach to propagators has become so important in modern quantum theories, it seems worthwhile to compare the *intuitive picture of quantum fluctuations* underlying both approaches. We deem the group-theoretic result stated in (3.14) as something deeply rooted in the quantum formalism. For this reason, we want to discuss finally a possible physical significance of this result, although in a rough, purely intuitive manner.

According to the Feynman approach, one visualizes the system as evolving from configuration spacetime point $x$ to point $x'$ along *all conceivable piecewise continuous curves* connecting these two events, whether these curves are consistent with the equations of motion or not. (In fact, a Feynman path integral is a clever device for summing up all the contributions made by 'quantum fluctuations' and thus obtaining the desired propagator.) On a purely intuitive level, however, one wonders if the Feynman picture is pushing too far the formal analogy of the effects of quantum fluctuations with some kind of 'Brownian motion.' Indeed, at this level of thought it would seem more "realistic" to think that the system evolves permanently under given

effective forces and therefore every quantum fluctuation makes it jump from one allowable configuration world line into another.

Given a configuration event $x = (t, \mathbf{x}) \in X$, in classical mechanics there are infinitely many world lines [with suitable parameters $(\alpha, \beta)$] which go through $x$. Hence, given two events $x$ and $x'$ there is in general *one*, and only one, curve in $X$ that goes through these two points, owing to the equations of motion. (We may here disregard some exceptional cases to this rule.) Therefore, given a particular world line $w_x$ through $x$, the classical probability of finding the system at $x'$ is either *zero* or *one* in usual circumstances. In quantum mechanics, however, there is a continuous probability amplitude for the system to go *freely* from $x$ to $x'$, whatever its state may be at $x$. This is due to spontaneous quantum fluctuations arising from the environment; as we know, no interaction with an apparatus is the 'cause' of this effect.

So, we can say that the main difference between the Feynman and the quantum-kinematic "pictorial" interpretation of this feature is the following. According to quantum kinematics, the system evolves permanently under its own law of force, but the quantum fluctuations (schematically denoted as stochastic events $x_1, x_2, x_3, \ldots$) derail the system, which thus jumps from one allowable world line into another. These effects may be mathematically described, in a *completely consistent manner*, as a 'mysterious' physical action of the adequate symmetry group on the states of the system. In fact, *each quantum fluctuation of a free system corresponds to a symmetry operation*, since it changes the state, but does not change the very physical nature of the system. In fancy language, one could claim that we do not know really what quantum fluctuations *are*; but we know what they *do*. In other words, because of the simultaneous local presence of both mechanical entities (i.e., the continuous applied forces and the permanent quantum fluctuations) the system can go from $x$ to $x'$ following any piecewise continuous curve $(x \to x_1 \to x_2 \to \cdots \to x')$ whose segments belong to the *admissible* world lines. The sum of the contributions made by all *physically fluctuating* world lines would then be given precisely by equation (3.14). Notwithstanding the fact that the Feynman construct for $\langle x | x' \rangle$ is not strictly *deductive* (as is the quantum kinematic one), both descriptions must give the same answer. This may be so because in the standard path-integral approach one introduces a suitable normalizing measure which lessens the 'nonphysical contributions' present in the functional integral.

## 4. CONCLUDING REMARKS

We have sketched a procedure by which non-Abelian quantum kinematics can produce a meaningful approach to quantum dynamics. This endeavor has been achieved under the special perspective afforded by Feynman's

postulate concerning the local phase of a Lagrangian system, as well as with the Feynman–Hibbs idea of using physical world-line segments instead of arbitrary straight lines for defining the subsets of paths contributing to the propagation kernel. Bear in mind, however, that equation (3.14) is the *fundamental result* in quantum kinematics. Equation (3.14) is valid even if the complete symmetry group $G$ has no *genuine ray extensions* by $U(1)$. In this case, the phase function $\varphi(x; q)$ is a gauge artifact which one may set equal to zero, because two-cocycles are just coboundaries for groups of such a kind. (As is well known, this is the case for the Poincaré group, which is one of the most conspicuous examples of such a group.) Nonetheless, transition probability amplitudes for irreducible configuration states are still given by equation (3.14) [with $\varphi(x; q) = 0$] even in such a case.

This is not the place to go into all the mathematical intricacies of the quantum kinematic approach to dynamics. (Neither was that the spirit of this paper.) However, here we shall make some remarks concerning the physical possibilities of the sketched formalism.

The main purpose in this trend of ideas would be to detach the quantum formalism from the classical formalism as far as possible, in order to obtain a self-contained quantum theory with its own *syntactic* and *semantic* rules. As we have seen, one can develop a new *group-theoretic quantization method*, in which the complete symmetry structure (associated with conservation laws and superselection rules) dictates the quantum rule, from which interesting quantum models can be expected to obtain. In this state of affairs, whether the resulting models admit a reasonable classical analog or not depends on the nature of things. The following points must be underlined concerning this issue. (a) Such a theory would correspond to a 'geometrization' of quantum dynamics, for it should stem exclusively from the assumed symmetries of the system. (b) The standard formulation of quantum mechanics could be recovered herein; as a matter of fact, it should appear as a particular case of the new generalized theory. (c) The new formulation uses a group-theoretic 'quantization' procedure (which is indeed the only notion that has been essentially at stake in this study).

Furthermore, the attained formalism is intrinsically 'relativistic' (in a rather general sense). In fact, it must be borne in mind that quantum kinematics is quite independent of the physical meaning attached to the variables $(x^1, \ldots, x^n)$ used in $X$ as long as these are 'preferred variables' for the action of $G$. The $x$'s correspond to both the independent and the dependent variables of the theory; moreover, in the previous context, some of the alluded to $x$'s may be *fields* defined in a spacetime arena. Nevertheless, it is not necessary to specify whether $X$ corresponds to an 'external' or an 'internal' configuration space in order for the formalism to hold. However, if the subspace of the independent variables contained in $X$ is an 'external' space, the quantum

kinematic approach to dynamics incorporates in a rather natural way the 'special relativity theory' (Mariwalla, 1975) attached to the given action in $X$ of the complete symmetry group of the system.

Note in particular that there is no need for the configuration variables to be Cartesian coordinates. In this fashion, the traditional dictum that 'one must quantize only in rectangular coordinates' (as required, in fact, by the usual Heisenberg commutation relations) is superseded in this theory. Such a proviso is certainly not a satisfactory state of affairs; one should not have so basic a criterion of quantization stated in such a strong coordinate-dependent manner. As an important related fact, let us also recall that the 'canonical quantization rule' is more formally apparent than effectively valid (Komar, 1971). It seems that non-Abelian group quantization offers a possible way out to this dilemma of contemporary quantum physics.

The interest of the quantization problem is not purely academic, of course, for it has many important consequences. As a case in point, which was already mentioned in the Introduction, here one faces one of the insurmountable facts that make quantization of general relativity theory so difficult; canonical quantization, as it stands, forbids the use of the principle of general covariance, while, on the other hand, general relativistic models of curved spacetime forbid the use of globally Cartesian coordinates. Certainly, this fundamental problem of quantum kinematics has remained unsolved, notwithstanding the interesting approaches to functional integration over geometries (Mottola, 1995) or to path integrals on homogeneous manifolds (Marinov, 1995), or even to time-displaced interactions to implement path-integral quantization (Schulman, 1995), which have been reported recently.

Indeed, much work has been devoted during recent years to developing a suitable quantum kinematic theory for curved spacetime, using Hartle's *sum-over-histories* generalized quantum mechanics (Gell-Mann and Hartle, 1992; Whelan, 1994), which seems to be particularly powerful for building models of quantum cosmology (Halliwell and Ortiz, 1993). As a matter of fact, there are several formal aspects of non-Abelian quantum kinematics which seem to appear also in the Hartle formalism (Omnès, 1992). There are significant differences, however. For instance, as far as we can see, according to the present quantum kinematic approach, one could identify $X$ with curved spacetime and a set of metric fields. In this way, the *symmetries* of the metric field $g_{\mu\nu}(x)$ would play the main role in a quantum theory of curved spacetime. We did not touch on this subject in this paper, of course, but in our opinion it deserves some attention.

Finally, we would also like to add the following remark. Once we are in possession of the group-quantization technique, we can go the other way around and consider the quantum kinematic theory of a *given physically relevant Lie group* as a *complete* dynamical theory. Thus, it seems reasonable

to expect that group quantization of the Poincaré group, by itself, could produce interesting quantum field models for *all* those elementary physical systems for which the familiar realization of Poincaré transformations in Minkowski space-time could be considered (by definition) as a *complete action* of the group. [In this sense, see, for instance, the recent work of Navarro *et al.* (1996), which comes very close to this idea.]

We thus finish our 'heuristic approach' to non-Abelian quantum kinematics. Needless to say, in the physical motivation presented in this paper, we do not pretend to have solved a theoretical problem in a rigorous manner. Rather, what has been done here is to try to pose some *fundamental questions on symmetry and dynamics*, as briefly and reasonably as possible, which lead to a serious and intriguing problem. As an extra bonus, a framework emerges from this study in order to perform the *systematic* group-theoretic steps which could afford an essentially new formulation of quantum mechanics. At least, we hope, the reader may now share our conviction that non-Abelian quantum kinematics is an important and timely issue.

We have decided to publish these ideas as they stand, since they afford a consistent group-theoretic framework for quantum physics, which seems promising. They are worth much further research by people interested in the fundamental role of symmetry in physics.

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